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A model of the universe with a variable G and a decaying vacuum energy

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Abstract : A geometrically closed but ever-expanding R-W model of the universe admitting a contracted Ricci-collineation along the fluid flow vector with variables G and Λ has been presented. The model evolves from rest from a non-singular hot origin with maximum radiation and vacuum energy densities and a minimum (non-zero) gravitational coupling G which increases and vacuum decays with expansion in a manner consistent with the conservation of energy momentum tensor of matter content The estimates of the present values of various cosmological parameters have been calculated.

Keywords : Robertson-Walker metric, family of contracted Ricci-collineations, cosmological parameters

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1. Introduction

In recent times, the cosmological constant Λ has interested theoreticians and observers for various reasons [1]. The non-trivial role of vacuum in the early universe generates a Λ term in the Einstein field equations that leads to the inflationary phase [2]. The inflationary cosmology postulates that during an early exponential phase, the vacuum energy was a large cosmological constant. Therefore, in view of the smallness of the cosmological constant observed at present, it is natural to assume that the cosmological constant Λ is a variable dynamic degree of freedom which being initially very large, relaxes to its small present value in an expanding universe. The idea of a dynamically decaying cosmological constant with cosmic expansion has been considered by several authors in the past few years [3–6].

As the Newtonian constant of gravity G plays the role of a coupling constant between geometry and matter in the Einstein field equations, it appears natural to look at this constant as a function of time in an evolving universe. There are many extensions of Einstein's theory of gravitation in which G is taken to vary with time [7]. Recently, it has been proposed to link the variation of G with that of the cosmological constant Λ leaving the form of the field equations unchanged and preserving the conservation of the energy-momentum tensor of the matter content [4]. However, this approach is non-covariant but it is worth studying because it may be a limit of some higher dimensional fully covariant theory.

In the present paper, a time-varying cosmological constant $\Lambda(t)$ representing the energy density of vacuum and a time-dependent gravitational coupling $G(t)$ have been considered in the framework of general relativity. A special case of the Robertson-Walker space-time admitting a contracted Ricci-collineation along the fluid flow vector is considered and thereby a model of the universe is investigated. The resulting model, being geometrically closed ($k = 1$), is ever-expanding and evolves from rest from a non-singular hot origin with maximum (finite) energy density and temperature and a small minimum (non-zero) gravitational coupling G. The decaying vacuum boosts G at a fast rate in the early universe and then comparatively slowly in the present phase of evolution. The model becomes globally causally connected in the early vacuum dominated phase itself. We also find the predicted values of some of the cosmological parameters.

2. The field equations

The universe is assumed to be filled with a distribution of matter represented by the energy momentum tensor of a perfect fluid

$$T_{ij} = (\rho + p)v_i v_j + p g_{ij}, \text{ (in the units with } c = 1), \tag{2.1}$$

where ρ is the energy density of the cosmic matter and p is its pressure. The geometry of an isotropic, homogeneous universe is described by the Robertson-Walker line element

$$ds^2 = -dt^2 + R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right\} \tag{2.2}$$

characterized by its scale factor $R(t)$ and the curvature parameter $k(=\pm 1, 0)$.

In the Gaussian normal coordinates of this metric, the fluid flow vector v^i is the normalized time-like eigenvector of the T_{ij}.

The Einstein field equation, with time dependent G and Λ, can be written as

$$R_{ij} - \frac{1}{2} R^h_h g_{ij} = -8\pi G(t)\left[T_{ij} - \frac{\Lambda(t)}{8\pi G(t)} g_{ij}\right]. \tag{2.3}$$

Here $-\dfrac{\Lambda}{8\pi G}g_{ij}$ is the energy momentum tensor of vacuum with its energy density ρ_v and homogeneous, isotropic pressure p_v satisfying the equation of state

$$p_v = -\rho_v = -\frac{\Lambda}{8\pi G}, \tag{2.4}$$

though other interpretations of the cosmological term also exist in the literature [8]. Thus, the total energy momentum tensor of the universe due to the presence of matter and vacuum can be written as

$$T_{ij}^{(\mathrm{tot})} \equiv T_{ij} + T_{ij}^{(v)} = (\rho_t + p_t)\, v_i v_j + p_t g_{ij}, \tag{2.5}$$

where $\rho_t = \rho + \rho_v$ and $p_t = p + p_v$. In the context of (2.1), (2.2) and (2.4), the field equation (2.3) yields

$$-\frac{\ddot{R}}{R} = \frac{4\pi}{3} G(t)\,(\rho + 3p - 2\rho_v) \equiv \frac{4\pi}{3} G(t)\,(\rho_t + 3p_t), \tag{2.6}$$

and

$$\frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = \frac{8\pi}{3} G(t)\,(\rho + \rho_v) \equiv \frac{8\pi}{3} G(t)\,\rho_t. \tag{2.7}$$

Symmetry methods of obtaining conservation laws in general relativity are well known. Collinson [9] has obtained a conservation law generator of the form

$$\left(R_m^j\, \eta^m\right)_{;\,j} = 0 \tag{2.8}$$

if the space-time admits a Ricci-collineation along the vector field η^i. Green *et al* [10] have discussed the symmetries of the Robertson-Walker metric belonging to the family of contracted-Ricci-collineations and the related conservation law generators by considering the symmetry vector proportional to the fluid flow vector. Out of the three possible choices of $\pounds R_{ij}$ studied by them, the one having

$$\pounds R_{ij} = \alpha\left(R_{ij} - \frac{1}{4} R_h^h\, g_{ij}\right) \tag{2.9}$$

leads to a symmetry vector $\eta^i = \psi v^i$ given by

$$\psi = \frac{C}{\ddot{R}k^{2}}, \quad C = \text{constant}, \tag{2.10}$$

without imposing any further restriction on the class of space-times.

Here, we further restrict the symmetry vector and the class of space-times by taking

$$\ddot{R}R^2 = \text{constant}, \tag{2.11}$$

which, from the field equation (2.6), leads to a conservation expression of the form

$$G(t)\,(\rho + 3p - 2\rho_v)R^3 \;=\; \text{constant} \;=\; A \text{ (say)}, \tag{2.12}$$

which in some sense may be interpreted as conservation of the total active gravitational mass of the universe.

It is clear from eq. (2.12) that for a decaying vacuum energy, the constant A may assume three different values (negative, zero and positive) in the three different phases of evolution depending upon the relative dominance of matter and vacuum. In the early vacuum dominated phase, $A < 0$. When vacuum reduces and balances its material counterpart, probably in the rest mass creation regime, $A = 0$. When vacuum further decays and gets dominated by matter, $A > 0$. These possible changes in A may be a result of two-phase transitions occurred in the early universe (The idea of two-phase transitions in the early universe, agrees with unified gauge field theories which indicate that the phase transitions are expected to have occurred at $T \simeq 10^{15}$ GeV and at $T \simeq 10^{2}$ GeV) which also accounts for the changes in $\dot{R}$ as in clear from eq. (2.11), which may be written as

$$R \;=\; -\,\frac{4\pi A}{3R^2}. \tag{2.13}$$

On integration, this gives

$$\dot{R}^2 \;=\; \frac{8\pi A}{3R} + B, \quad B \;=\; \text{constant}, \tag{2.14}$$

supplying the time-variation of $R(t)$ independently of the value of k. Eqs. (2.7) and (2.14) give

$$\rho + \rho_v \;=\; \frac{A}{GR^3} + \frac{3(B+k)}{8\pi GR^2}, \tag{2.15}$$

which, taken together with (2.12), obtains

$$p - \rho_v \;=\; -\,\frac{(B+k)}{8\pi GR^2}. \tag{2.16}$$

An elimination of $\ddot{R}$ between eqs. (2.6) and (2.7) yields

$$\dot{\rho} + 3(\rho+p)\,\frac{\dot{R}}{R} + \rho_v + (\rho+\rho_v)\,\frac{\dot{G}}{G} \;=\; 0, \tag{2.17}$$

which with the assumption of the law of energy-momentum conservation $\left(T^{ij}_{;j} = 0\right)$ given by

$$\dot{\rho} + 3(\rho+p)\,\dot{R}/R \;=\; 0, \tag{2.18}$$

obtains

$$\frac{\dot{G}}{G} \quad \text{[illegible]} \quad (\rho+\rho_v), \tag{2.19}$$

indicating that G increases or decreases according as ρ_v respectively decreases or increases.

As eqs. (2.15) – (2.19) supply only three independent equations in four unknowns, ρ, p, ρ_v and G, an extra equation is needed to solve the system uniquely. This is supplied by the equation of state of the matter content :

$$p = w\rho, \qquad w = \text{constant}. \tag{2.20}$$

By the use of this, eqs. (2.15) and (2.16) obtain

$$GM = \frac{\pi(B+k)}{2(1+w)}\left[\frac{4\pi A}{(B+k)} + R\right. \tag{2.21}$$

which is very similar to the conservation relation

$$GM = R, \tag{2.22}$$

obtained by a number of authors mostly by invoking Mach's principle [11]. Here, $M = 2\pi^2 R^3 \rho$ is the mass of the observable universe with radius R.

We conclude this section by noting another consequence of the assumption (2.13). As $(-\ddot{R}/R)$ is the Gaussian curvature of the two dimensional surface specified by varying r and t, keeping θ and ϕ constant in (2.2) and may be considered as the curvature of the homogeneous, isotropic space-time represented by (2.2), eq. (2.13) thus indicates that the curvature of the space-time continuously rolls down as universe expands. This consequently transforms the space-time from a state of large curvature to a state of flatness as $t \rightarrow \infty$.

The model is now completely specified in dynamical structure. Eqs. (2.12) – (2.21) determine the physical content and the evolution of the model. We further investigate its cosmological consequences in the following sections starting with the early radiation dominated phase.

3. The very early universe

We choose the initial time $t = 0$, the one when $\dot{R} = 0$, $R = R_o$, $\rho = \rho_o$, $G = G_o$ and $\rho_v = \rho_{vo}$. Eq. (2.7) then obtains $G_o = \dfrac{3k}{8\pi(\rho_o + \rho_{vo})R_o^2}$. As G_o, ρ_o and ρ_{vo}, all are positive quantities, we have $k = 1$ so that $G_o = \dfrac{3}{8\pi(\rho_o + \rho_{vo})R_o^2}$.

Taking $w = 1/3$ in this phase, eqs. (2.12) and (2.14) suggest that

$$A = -\frac{3}{4\pi}\frac{(\rho_{vo}-\rho_o)}{(\rho_{vo}+\rho_o)}R_o \text{ and } B = 2\frac{(\rho_{vo}-\rho_o)}{(\rho_{vo}+\rho_o)}. \tag{3.1}$$

For the value of A given in (3.1), eq. (2.13) reduces to

$$\ddot{R} = \frac{(\rho_{vo} - \rho_o)}{(\rho_{vo} + \rho_o)}\cdot\frac{R_o}{R^2}, \tag{3.2}$$

which suggests that ρ_{vo} must be greater than ρ_o for expansion to take place. Hence $A < 0$, B is a positive dimensionless number and $\ddot{R} > 0$ in this era. This indicates the existence of some general (*i.e.* non-exponential) type of inflation in the early universe. With $p = \frac{\rho}{3}$, eq. (2.18) yields the solution

$$\rho R^4 = \text{constant} = \rho_o R_o^4. \tag{3.3}$$

By the use of this, eqs. (2.15) and (2.16) give

$$G = \frac{G_o R}{2\rho_o R_o^2}\left[3\rho_{vo}(R-R_o) - \rho_o(R-3R_o)\right], \tag{3.4}$$

and

$$\rho_v = \frac{\rho_o R_o^4}{R^4}\left[\frac{\rho_{vo}(3R-R_o) - \rho_o(R-R_o)}{3\rho_{vo}(R-R_o) - \rho_o(R-3R_o)}\right]. \tag{3.5}$$

Assuming that the radiation temperature T is related to its density ρ by

$$\rho = \frac{\pi^2}{15}T^4 \quad \text{(in suitable units)}, \tag{3.6}$$

the variation of T is obtained as

$$T = \left[\frac{15\rho_o R_o^4}{\pi^2 R^4}\right]^{1/4}, \tag{3.7}$$

which suggests that T is maximum at $t = 0$ with $T_{\max} = \left[\frac{15\rho_o}{\pi^2}\right]^{\frac{1}{4}}$. With (3.1), eq. (2.14) reduces to

$$\dot{R}^2 = 2\,\frac{(\rho_{vo}-\rho_o)}{(\rho_{vo}+\rho_o)}\left(1 - \frac{R_o}{R}\right), \tag{3.8}$$

which may be solved as

$$R = R_o \operatorname{cosec}^2\psi,$$

$$t = \frac{R_o}{\sqrt{2}}\sqrt{\frac{(\rho_{vo}+\rho_o)}{(\rho_{vo}-\rho_o)}}\left[\operatorname{cosec}\psi \cot\psi + \ln\cot\frac{\psi}{2}\right] \tag{3.9}$$

As the model is geometrically closed ($k = 1$), it is possible to determine the time $t = t_c$ when the whole universe becomes causally connected. This is given by [5,12]

$$\int_o^{t_{cau}} \frac{dt}{R(t)} = \int_o^1 \frac{dr}{\sqrt{1-r^2}} = \pi/2, \tag{3.10}$$

which by the use of (3.9), obtains

$$\left.\begin{aligned} t_{cau} &= \frac{R_o}{\sqrt{2}}\sqrt{\frac{(\rho_{vo}+\rho_0)}{(\rho_{vo}-\rho_0)}}\left[\operatorname{cosec}\psi_{cau}\cot\psi_{cau} + \ln\cot\left(\psi_{cau}/2\right)\right], \\ \text{with } \psi_{cau} &= 2\tan^{-1}\left[\exp\left\{-\frac{\pi}{2\sqrt{2}}\sqrt{\frac{\rho_{vo}-\rho_0}{\rho_{vo}+\rho_0}}\right\}\right]. \end{aligned}\right\} \tag{3.11}$$

As there are two phase transitions between the early universe scenario and the present phase of evolution, discussed in the following section, the early evolution of the model discussed in this section is altogether isolated from the present universe and it may not be possible to extrapolate in time towards the initial state from the present universe solutions. However, a comparison with the standard model is always possible. In this regard, we consider $T_l = 4000$ K, the temperature at the time of last scattering which was achieved at about 4×10^5 yrs in the standard model (around when the universe started to become matter dominated [13]) and try to find out when it was achieved in our model and what were the values of other parameters at that time.

As the present 2.7 K ($= T_p$) cosmic microwave background radiation is a relic of the early hot universe, it is natural to expect that its temperature which was high enough in the early universe, relaxed to its present value according to eq. (3.7) which indicates that

$$T_l R_l = T_p R_p = \left(\frac{15\rho_o R_o^4}{\pi^2}\right)^{\frac{1}{4}}. \tag{3.12}$$

Thus, for $T_l = 4000$ K, $T_p = 2.7$ K and the later estimate of the present 'radius' of the universe $R_p = 2.47 \times 10^{42}$ GeV^{-1}, gives

$$R_l \simeq 1.7\times 10^{39} \text{ GeV}^{-1}. \tag{3.13}$$

Eq. (3.12) then gives

$$\rho_o R_o^4 \simeq 7.2\times 10^{118}. \tag{3.14}$$

Eq. (3.4) now yields

$$\frac{x(3-y) + 3(y-1)}{x(1+y)} = \left(4.4 \times 10^{41} \text{ GeV}^2\right) G_l = a \text{ (say)} \tag{3.15}$$

giving
$$y = \frac{(3-a)x-3}{(1+a)x-3}, \tag{3.16}$$

where $x = R_l/R_o$ and $y = \rho_o/\rho_{vo}$. Eq. (3.9) then obtains

$$\frac{1}{x}\sqrt{\frac{2x-3}{x(a-1)}}\left[\sqrt{x(x-1)} + \ln\left(\sqrt{x}+\sqrt{x-1}\right)\right] = \frac{\sqrt{2}}{R_l}\, t_l. \tag{3.17}$$

As one naively expects x to be a sufficiently large number ($R_l >> R_o$), eqs. (3.16) and (3.17) respectively reduce to

$$y \simeq \frac{3-a}{1+a}, \tag{3.18}$$

$$t_l \simeq \frac{R_l}{\sqrt{(a-1)}}. \tag{3.19}$$

As $0 < y < 1$, eq. (3.18) suggests that $1 < a < 3$. This gives

$$2.3\times10^{-42}\ \mathrm{GeV}^{-2} < G_l < 6.9\times10^{-42}\ \mathrm{GeV}^{-2}. \tag{3.20}$$

Thus, one can safely choose $G_l \simeq 10^{-42}\ \mathrm{GeV}^{-2}$ with a moderate value of a *i.e.*, $a \simeq 2$. Eqs. (3.18) and (3.19) therefore, respectively give

$$\rho_0 \simeq \frac{1}{3}\rho_{vo}, \tag{3.21}$$

$$t_l \simeq R_l \simeq 3.5\times10^{7}\ \mathrm{yrs}. \tag{3.22}$$

Thus $T_l = 4000$ K is achieved in our model, a little later than in the standard model.

By the use of (3.21), eq. (3.11) gives

$$t_{\mathrm{cau}} \simeq 1.9\, R_o. \tag{3.23}$$

In order to have an estimate of the initial radius R_o, we associate it with the limiting time to which the equations of the early universe can be pushed back. The limit arises when the classical general relativity is no longer reliable and one must resort to quantum gravity. The limit in time is the so called Planck time $\simeq 5\times10^{-44}\ s$ [14] (The terminology, in the present context, is no longer appropriate as 'Planck time' also varies with time for $G = G(t)$). The vacuum energy density of quantum fields at this time is given as $2\times10^{71}\ \mathrm{GeV}^4$ by Weinberg [1]. We then readily assume $\rho_{vo} = 2\times10^{71}\ \mathrm{GeV}^4$ giving $\rho_0 \simeq 6.7\times10^{70}\ \mathrm{GeV}^4$ which is in good agreement with the upper limit of the energy density as $10^{74}\ \mathrm{GeV}^4$ that matter can have [15].

Other initial parameters are then immediately obtained :

$$T_{\max} \simeq 5.6\times10^{17}\ \mathrm{GeV},\quad R_o \simeq 1\times10^{12}\ \mathrm{GeV}^{-1}$$

and $$G_o \simeq 4.3\times10^{-2}\ \mathrm{GeV}^{-2}.\ \text{Hence}\ t_{\mathrm{cau}} \simeq 1.3\times10^{-12}\ s.$$

4. The present universe

Observations [6,16] indicate that the present cosmic energy density ρ_p is very close to its corresponding critical value $\rho_{cp} \equiv \frac{3}{8\pi G_p} H_p^2$, where H_p is the present value of the Hubble parameter. Taking $w = 0$ in the present phase of evolution, eqs. (2.14), (2.15) and (2.16) then suggest that $B = 2$ and $\rho_{vp} = \frac{3}{8\pi G_p R_p^2}$. The value of A in this era, is then obtained from eq. (2.12) as

$$A = \left(G_p \rho_p R_p^2 - \frac{3}{4\pi}\right) R_p. \tag{4.1}$$

For $p = 0$, eq. (2.18) leads to

$$\rho R^3 = \text{constant} = \rho_p R_p^3 \equiv E_p. \tag{4.2}$$

By the use of this, eqs. (2.15) and (2.16) in this era, obtain

$$G = G_p + \frac{3(R - R_p)}{4\pi E_p} \tag{4.3}$$

and

$$\rho_v = \frac{3 E_p}{2R^2 \left\{4\pi G_p E_p + 3(R - R_p)\right\}}. \tag{4.4}$$

Eq. (4.3) indicates that the rate of increase of G in the present phase has slowed down from that in the early universe. Eqs. (3.5) and (4.4) indicate that the energy density of vacuum ρ_v, akin to energy density of matter ρ, varies differently in the different phases of evolution.

From eq. (4.3), we have

$$\frac{(\dot{G})_p}{G_p} = \frac{3H_p}{4\pi G_p \rho_p R_p^2}. \tag{4.5}$$

For the observational values

$$\frac{(\dot{G})_p}{G_p} = 10^{-11}\ \text{yr}^{-1},\ H_p = 7.5 \times 10^{-11}\ \text{yr}^{-1},\ G_p = 6.7 \times 10^{-39}\ (\text{GeV})^{-2}$$

and hence $$\rho_p = \rho_{cp} \left(\equiv \frac{3H_p^2}{8\pi G_p}\right) = 4.4 \times 10^{-47}\ (\text{GeV})^4,$$

eq (4.5) gives

$$R_p = 2.47 \times 10^{42}\ (\text{GeV})^{-1}. \tag{4.6}$$

Eq. (4.1) then supplies an estimate of A in the present phase as

$$A = 3.84 \times 10^{42} \ (\mathrm{GeV})^{-1}, \tag{4.7}$$

which with (2.13), suggests that $\dot{R} < 0$ in this era.

The present value of the cosmological constant may be obtained from eqs. (2.4) and (4.4) as

$$\Lambda_p = 4.9 \times 10^{-85} \ (\mathrm{GeV})^2, \tag{4.8}$$

which is well within the upper limit of Λ_p found as 10^{-82} $(\mathrm{GeV})^2$ from the cosmological observations [17].

The present value of the deceleration parameter $q \equiv -\frac{R\ddot{R}}{\dot{R}^2}$ may be obtained from eqs. (2.13) and (2.14) which yield

$$q = \frac{1}{2}\left[1 + \frac{3R}{4\pi A}\right]^{-1}, \tag{4.9}$$

giving $\quad q_p = 0.43. \qquad (4.10)$

The time dependence of the scale factor R in this era, is obtained from eq. (2.14) as

$$\begin{aligned} R &= \frac{2\pi A}{3}\left(\cosh 2\psi - 1\right), \\ t &= \frac{\sqrt{2}\pi A}{3}\left(\sinh 2\psi - 2\psi\right) + D. \end{aligned} \tag{4.11}$$

where A is given by (4.7) and D is a constant of integration. We thus, observe that $R \to \infty$ as $t \to \infty$ although $k = 1$. This is a significant deviation from the standard model.

Finally, we find an estimate of the age of the universe t_p by assuming that the early phase of evolution is very small compared to the maximum matter dominated part of the evolution. Eq. (4.11) by the use of (4.6) and (4.7), then gives

$$t_p \simeq 9.14 \times 10^9 \text{ years}.$$

This is approximately the same as in the standard model and looks significantly smaller than $12 - 18 \times 10^9$ yrs, the age of the globular clusters which are among the oldest objects in the galaxy. However, the comparison of the age of the universe from observations with that from models, is somewhat inconclusive [13,18]. Moreover, there is still an uncertainty in the value of H_p which varies by a factor 2; $50 \leq H_p \leq 100$ km s^{-1} Mpc^{-1} rendering the age of the universe in our model as $6.8 - 13.7 \times 10^9$ yrs.

5. Concluding remarks

We consider a Robertson-Walker space-time admitting a contracted Ricci-collineation along the fluid flow vector and find that the quantity $G\,(\rho + 3p - 2\rho_v)\,R^3$ is conserved. With this, we investigate a cosmological model wherein the cosmological constant Λ, representing the energy density of vacuum, and the gravitational coupling G are taken to vary with time in a way which conserves the energy momentum tensor of the matter content. The resulting model is geometrically closed though it is ever-expanding and evolves from rest from a non-singular hot origin with maximum radiation and vacuum energy densities and a small (minimum) gravitational coupling G and soon becomes globally causally connected in the early vacuum dominated era which is endowed with some general type of inflation. The decreasing vacuum energy density promotes G, first at a very fast rate in the early phase of evolution and then comparatively slowly in the present phase of evolution.

Specifying the present value of the matter energy density ρ as equal to its corresponding critical value ρ_c and using the frequently quoted values of the Hubble parameter and the fractional rate of change of the gravitational 'constant' $\dot{G}/G$, we calculate the estimates of the present values of the scale factor, the cosmological constant, the deceleration parameter and the age of the universe. The present value of the cosmological constant is well within the experimental limit which is remarkable since its initial value in the model is very large.

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